An Informal Introduction to HoTT via Synthetic Homotopy Theory

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April 09, 2025

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About Me

Research Interests

- Displayed Univalent Reflexive Graphs in Cubical Type Theory (w/ Streicher, Buchholtz)
- Primitive Recursive Dependent Type Theory [Buchholtz and Schipp von Branitz, 2024]
- Geometric Type Theory (w/ Buchholtz)

Homotopy Type Theory

Motivation

- Computer verifiable mathematical foundation
  - MLTT: 1972
  - Groupoid Interpretation: Hofman, Streicher 2002
  - Univalence axiom, homotopical models: Voevoedsky, Awodey, Warren 2005
- Higher generality due to Grothendieck ∞-topos semantics
- Technical simplicity
- Unification of logic and structure
- Independence of models of ∞-categories
- Structure Identity Principle

Homotopy Type Theory

Dictionary

Notation	Name	Logic	Set Theory	Geometry
A	type	proposition	set	space
$a : A$	term	proof	element	point
$f : A \to B$	function	implication	function	continuous map
$\mathcal{U}$	universe	-	Groth. universe	space of small spaces
$B : A \to \mathcal{U}$	type family	family of propositions	family of sets	fibration
$\Sigma_{a:A} B(a)$	dep. pair tp.	exist. quant.	disj. union	total space
$\Pi_{a:A} B(a)$	dep. function tp.	univ. quant.	indexed prod.	space of sections
$(x =_A y)$	identity tp.	equality	equality	path fibration
$p : (x =_A y)$	identification	-	-	path

Homotopy Type Theory

Dependent Function Types

Let $\Gamma, x : A \vdash B(x)$ type be a type in context. Then we can form

$$\prod_{x:A} B(x)$$

With introduction rule

$$\frac{\Gamma, x : A \vdash f(x) : B(x)}{\Gamma \vdash \lambda(x : A).f(x) : \prod_{x:A} B(x)},$$

elimination rule

$$\frac{\Gamma \vdash f : \prod_{x:A} B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B(a)}$$

such that

$$\lambda((x : A).f(x))(a) \equiv a$$

and

$$\lambda(x : A).f(x) \equiv f.$$

Semantically, the introduction rule is the right adjoint to pullback along $\Gamma.A \to \Gamma$.

Similarly, terms of type

$$\sum_{x:A} B(x)$$

are exactly given by pairs of $a : A$ and $b : B(a)$:

$$\mathrm{ind}_{\Sigma} : \Big(\prod_{a:A} \prod_{b:B(x)} P(a,b)\Big) \to \prod_{z:\sum_{x:A} B(x)} P(z).$$

Semantically, the introduction rule is the left adjoint to pullback along $\Gamma.A \to \Gamma$.

Homotopy Type Theory

Identity Types: Formation and Introduction

Given $x, y : A$ we can form the inductive family

$$(x =_A y)$$

with constructor

$$\mathsf{refl}_x : (x =_A x).$$

Semantically, given an ∞-groupoid A, exponentiation with the simplicial interval

$$\Delta^0 + \Delta^0 \rightarrowtail \Delta^1 \xrightarrow[\sim]{} \Delta^0$$

Gives rise to a natural path space factorisation

$$A \overset{\mathsf{refl}}{\underset{\sim}{\rightarrowtail}} A^{\Delta^1} \twoheadrightarrow A \times A$$

of the diagonal.

Homotopy Type Theory

Identity Types: Elimination and Computation

The induction principle states that given a type family

$$x : A, y : A, p : (x =_A y) \vdash B(x, y, p) : \text{ type}$$

and a family of terms

$$a : A \vdash b(a) : B(a, a, \mathrm{refl}_a)$$

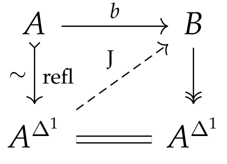
we have a proof

$$x : A, y : A, p : (x =_A y) \vdash \mathrm{J}_b(x, y, p) : B(x, y, p)$$

and satisfies the computation rule

$$\mathrm{J}_b(x, x, \mathrm{refl}_x) \equiv b(x)$$

Semantically,



We can define *path composition*

$$\_ \cdot \_ : \prod_{\{x,y,z:A\}} (x =_A y) \to ((y =_A z) \to (x =_A z))$$

$$\mathsf{refl}_x \cdot p :\equiv p$$

by path induction on the first path argument. Similarly, we obtain *path inversion*

$$\_^{-1} : \prod_{\{x,y,z:A\}} (x =_A y) \to (y =_A x)$$

$$\mathsf{refl}^{-1} :\equiv \mathsf{refl}.$$

The same idea works for associativity, MacLane pentagon, unit laws, etc.

Given $f : A \to B$ we have

$$\mathsf{ap}_f : \prod_{\{x,y:A\}} (x =_A y) \to (f(x) =_B f(y))$$
$$\mathsf{ap}_f(\mathsf{refl}_x) :\equiv \mathsf{refl}_{f(x)}$$

and

$$\mathsf{apConcat} : \prod_{\{x,y,z:A\}} \prod_{p:(x=_A y)} \prod_{q:(y=_A z)} (\mathsf{ap}_f(p \cdot q) =_{(f(x)=_B f(z))} \mathsf{ap}_f(p) \cdot \mathsf{ap}_f(q))$$

from assuming $p = \mathsf{refl}_x$ and using $\mathsf{refl}_x \cdot q = q$, as well as $\mathsf{ap}_f(\mathsf{refl}_x) \equiv \mathsf{refl}_{f(x)}$.

Homotopy Type Theory

Homotopies

Given $f, g : \prod_{a:A} B(a)$, we define the *type of homotopies*

$$f \sim g :\equiv \prod_{a:A} (f(a) =_{B(a)} g(a)).$$

A map $f : A \to B$ is an *equivalence*, if there are $g, h : B \to A$ such that $f \circ g \sim \mathrm{id}_B$ and $h \circ f \sim \mathrm{id}_A$.

Homotopy Type Theory

Univalent Universes

We assume the existence of *universes* encoding small types and which are closed under all type theoretic constructions.
A universe $\mathcal{U}$ is *univalent*, if for all $A, B : \mathcal{U}$, the natural map

$$\begin{gathered}(A =_{\mathcal{U}} B) \to (A \simeq B) \\ \mathrm{refl}_A \mapsto \mathrm{id}_A\end{gathered}$$

is an equivalence.
It is an *axiom* that all our universes are univalent. This is incompatible with the set model.

Homotopy Type Theory

Function Extensionality

Function extensionality states that for $f, g : \prod_{a:A} B(a)$ the canonical map

$$\begin{aligned}(f =_{\prod_{a:A} B(a)} g) &\to (f \sim g)\\ \mathrm{refl}_f &\mapsto \lambda(a : A).\mathrm{refl}_{f(a)}\end{aligned}$$

is an equivalence. It lets us prove universal properties such as

$$\big((A + B) \to X\big) \simeq \big((A \to X) \times (B \to X)\big).$$

Univalence implies function extensionality.

Synthetic Homotopy Theory

Homotopy Levels

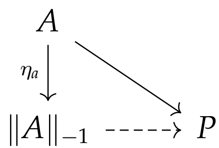
We have an inductive predicate

$$\begin{aligned}\mathrm{isTrunc} &: \mathrm{N}_{-2} \to \mathcal{U} \to \mathcal{U} \\ \mathrm{isTrunc}_{-2}(A) &:\equiv \sum_{c:A} \prod_{x:A} (c =_A x) \\ \mathrm{isTrunc}_{n+1}(A) &:\equiv \prod_{x,y:A} \mathrm{isTrunc}_n(x =_A y)\end{aligned}$$

expressing that A has no nontrivial homotopic information above level n. For the lower levels we prefer to use the following terminology

Truncation Level	Name
-2	contractible
-1	proposition
0	set
1	groupoid
...	...

Given a type A, its best approximation as an n-type is given by its *truncation* $\|A\|_n$ with the universal property



for an arbitrary n-type P.

HIGHER INDUCTIVE TYPES

Higher inductive types are inductive types which may additionally have path constructors. They guarantee the existence of truncations, e.g. the HIT X with the constructors

$$\eta : A \to X$$
$$\alpha : \prod_{x,y:X} (x =_X y)$$

satisfies the universal property of propositional truncation. HITs also yield homotopy colimits, e.g. the pushout of $f : A \to B$ and $g : A \to C$ is the HIT $B +_A C$ generated by

$$\mathrm{inl} : B \to B +_A C$$
$$\mathrm{inr} : C \to B +_A C$$
$$\mathrm{coh} : \prod_{a:A} (\mathrm{inl}(f(a)) = \mathrm{inr}(\mathrm{g}(\mathrm{a}))).$$

Synthetic Homotopy Theory

Connectedness

The *fiber* of a map $f : A \to B$ at $b : B$ is

$$\mathrm{fib}_f(b) :\equiv \sum_{a:A} (f(a) =_B b).$$

We say that f is n-connected, if all truncated fibers $\|\mathrm{fib}_f(b)\|_n$ are contractible.

Synthetic Homotopy Theory

Homotopy Groups

The *iterated loop space* of pointed types is defined as

$$\Omega : \mathbb{N} \to \sum_{X:U} X \to \sum_{X:U} X$$

$$\begin{aligned}
\Omega_0(A, a) &:\equiv (A, a) \\
\Omega_1(A, a) &:\equiv ((a =_A a), \mathrm{refl}_a) \\
\Omega_{n+1}(A, a) &:\equiv \Omega_1(\Omega_n(A, a))
\end{aligned}$$

The n-th *homotopy group* is

$$\pi_n(A, a) :\equiv \|\Omega_n(A, a)\|_n,$$

with unit $\eta(\mathrm{refl})$ and multiplication given by path concatenation.

Given a group G, its *Eilenberg-MacLane space* $\mathrm{K}(G,1)$ is the higher inductive type with constructors

$$\begin{aligned}
\star &: \mathrm{K}(G,1) \\
p &: G \to (\star = \star) \\
q &: (g,h : G) \to (p(gh) = p(g) \cdot p(h)) \\
\epsilon &: \mathrm{isTrunc}_1(\mathrm{K}(G,1)).
\end{aligned}$$

We inductively define

$$\mathrm{K}(G,n+1) :\equiv \|\Sigma \mathrm{K}(A,n)\|_{n+1},$$

where ΣX is the homotopy pushout of $1 \leftarrow X \to 1$. Then $K(G,n)$ is $(n-1)$-connected and n-truncated, and $\pi_n \mathrm{K}(G,n) \simeq G$.

As a special case, we can define S^1 as the HIT

$$\begin{aligned} \mathrm{base} &: S^1 \\ \mathrm{loop} &: (\mathrm{base} = \mathrm{base}) \end{aligned}$$

with universal property that the map

$$\begin{aligned} \prod_{z:S^1} P(z) &\to \sum_{x:P(\mathrm{base})} (y =^P_{\mathrm{loop}} y) \\ f &\mapsto (f(\mathrm{base}), \mathrm{apd}_f(\mathrm{loop})) \end{aligned}$$

is an equivalence.

A map $f : A \to B$ is ∞-*connected* if the following equivalent conditions are satisfied.

1. The induced maps on points and homotopy groups $\pi_k(f)$ are bijective for all $k \geq 0$.
2. For all $b : B$ and all $k \geq -2$, the truncated fiber $\|\mathrm{fib}_f(b)\|_k$ is contractible.

Whitehead's principle is the statement that every ∞-connected map is an equivalence. In the standard model of ∞-groupoids this statement corresponds to the classical Whitehead Theorem, but it fails to be true in non-hypercomplete ∞-toposes. However, the truncated variant is still true.

Current Research

- Define Semisimplicial Types
- Does HoTT eat itself?
- Computational Univalence
- Modalities
- Synthetic Mathematics
    - $(\infty, 1)$-category theory
    - Tait Computability
    - Geometric Type Theory

Resources

- Slides: `jsvb.xyz`
- Semantics: [Riehl, 2024]
- Syntax, Intuition: [Rijke, 2022, Univalent Foundations Program, 2013]
- Higher groups: [Buchholtz et al., 2018]

QUESTIONS?

Thank you!

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