Propositional Geometric Type Theory

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Geometric Type Theory

Setting & Idea

Kind of Space	Geometric Int. Lang	Full Int. Lang
$\mathrm{Top}_{\mathrm{sob}}$	prop. geom. logic	compl. Heyting alg.
Loc	prop. geom. logic	compl. Heyting alg.
Topos	geom. logic	MLTT
$\infty-$Topos	∞-geom. logic?	HoTT

- Toposes classify geometric theories $\mathbb{T}$:

$$\mathrm{Topos}(\mathcal{E}, [\mathbb{T}]) \cong \mathrm{Mod}_{\mathbb{T}}(\mathcal{E}).$$

- Geometric logic is incomplete, so we need to study models in all toposes.
- Some topos-valid constructions, such as Π-types, are not geometric/continuous, i.e. not preserved by inverse image functors.
- There are still toposes classifying arbitrary objects or maps, so geometric reasoning should suffice.
- This suggests treating toposes as types (cf. [Vic07]).

Geometric Type Theory

Motivation

- Recognition of geometric statements
- Transfer of results: *geometric* consequences of non-geometric statements are preserved
- Unification of external and internal perspective
- Unification of synthetic mathematics (SAG, SDG, STC)
- General treatment of modalities
- Recognition of classified geometric theories
- Synthetic Morita equivalences/bridges
- Definition of ∞-geometric logic
- Formalisation

There are lots of related ideas. None of them talk faithfully about *all* toposes, use the universal property of toposes as classifying spaces, and are an extension of HoTT.

- Topos-theoretic Multiverse [Ble]
- Multimodal Adjoint Type Theory [Shu23]
- Continuous Truth [Fou13]
- Abstract Stone Duality [Tay11]
- Synthetic Topology [Esc04]
- Synthetic Topos Theory [Uem]
- Arithmetic Type Theory [Vic08]

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Sketch of Intended Semantics

GTT should[3] be modeled by

$$\mathrm{Sh}_\infty(\mathrm{Topos}^1_{(2,1)}, \mathrm{J}_{\text{étale}})$$

with base type $\mathbf{O} = [\mathrm{FinSet}, \mathrm{Set}]$ classifying étale spaces (a.k.a. internal types)

$$\begin{array}{ccc} \mathbf{T}(X) & \longrightarrow & \dot{\mathbf{O}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{E} & \xrightarrow{X} & \mathbf{O} \end{array},$$

letting us recover the Sierpiǹski space

$$\mathbf{S} = \sum_{X:\mathbf{O}} \mathrm{isProp}(\mathbf{T}(X)).$$

[3] up to strictness and size issues, and a good choice of Grothendieck topology

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Intended Semantics

- Approximate above setting using semantics in the category

$$\mathrm{Sh}_0(\mathrm{Loc}, \mathrm{J}_{\mathrm{opencover}})$$

  of sheaves of sets on the category of localic toposes with the subcanonical open cover topology.
- The Sierpiǹski space classifies open subtypes:

$$\begin{array}{ccc} \mathbf{T}(p) & \longrightarrow & \mathbf{1} \\ \big\downarrow & \lrcorner & \big\downarrow \top \\ A & \xrightarrow{\;p\;} & \mathbf{S} \end{array} .$$

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Syntax

- Intensional MLTT
- Tarski-style universe

$$\frac{\Gamma \vdash p : \mathbf{S}}{\Gamma \vdash \mathbf{T}(p) \text{ type}}$$

- Bottom and top elements $\bot, \top : \mathsf{S}$ with $\mathbf{T}(\bot) \equiv \mathbf{0}$ and $\mathbf{T}(\top) \equiv \mathbf{1}$.
- Primitive binary conjunctions $\wedge : \mathsf{S} \to \mathsf{S} \to \mathsf{S}$
- Order relation $p \leq q :\equiv (p \wedge q =_{\mathbf{S}} p)$
- Meet-semilattice axioms

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Overt Discrete Spaces

- $f : A \to B$ is *open* if
$$f^* : (B \to \mathrm{S}) \to (A \to \mathrm{S})$$
has a left adjoint $f_!$.
- I is *overt* if $! : I \to \mathbf{1}$ is open, yielding
$$\bigvee : (I \to \mathbf{S}) \to \mathbf{S}.$$
- A is *discrete* if $\Delta : A \to A \times A$ is open
- Assume $\mathbf{N}$ is overt discrete, $\mathbf{S}$, $\mathbf{T}(p)$ overt.
- Being overt discrete is closed under positive type formers.
- $$(\mathbf{T}(p) \to \mathbf{S}) \to \sum_{q:\mathbf{S}} (q \leq p)$$
$$\varphi \mapsto p \wedge \bigvee_{x:\mathbf{T}(p)} \varphi(x)$$
is an equivalence.

For A and B overt discrete we should have *directed univalence*

$$(A \to B) \tilde{\to} \sum_{\gamma : \mathbf{S} \to \mathrm{ODisc}} (\gamma(\bot) = A) \times (\gamma(\top) = B)$$
$$f \mapsto \lambda p. \sum_{b:B} \mathbf{T}(p) \star \mathrm{fib}_f(b),$$

just like in Condensed Type Theory [Bar24, Com24].

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Current Work

- Characterise the topology of function spaces
- Use synthetic quasi-coherence and local choice
- Justify usability of our theory by proving (cf. [Hyl81])

$$\big((\mathbf{N} \to \mathbf{2}) \to \mathbf{2}\big) \simeq \mathbf{N}$$

- Extend simplicial aspects
- Extend to full Geometric Type Theory

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