Primitive Recursive Dependent Type Theory

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Primitive Recursion

REMINDER

The *basic primitive recursive functions* are constant functions, the successor function and projections of type $\mathbb{N}^n \to \mathbb{N}$. A *primitive recursive function* is obtained by finite applications of composition of the basic p.r. functions and the *primitive recursion operator*

$$\begin{aligned} \mathrm{primrec} : \mathbb{N} \to (\mathbb{N} \times \mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N} \\ \mathrm{primrec}(g, h, 0) &= g \\ \mathrm{primrec}(g, h, k+1) &= h(k, \mathrm{primrec}(g, h, k)). \end{aligned}$$

The *Ackermann function* $A : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ given by

$$\begin{aligned} A(0) &= (n \mapsto n+1) \\ A(m+1) &= \begin{cases} 0 & \mapsto A(m, 1) \\ n+1 & \mapsto A(m, A(m+1, n)) \end{cases} \end{aligned}$$

grows faster than any p.r. function. Defining it requires elimination into a function type.

Primitive Recursive Dependent Type Theory

Main Theorem

Let T be a dependent type theory with

- dependent pair types $\sum_{a:A} B(a)$ and function types $\prod_{a:A} B(a)$,
- inductive identity types,
- a universe U_0 closed under Σ- and identity types (but not Π-types),
- a U_0-small closed type N with the standard elimination principle for natural numbers

$$\frac{n : \mathrm{N} \vdash X(n) : \mathrm{U}_0 \qquad \vdash g : X(0) \qquad n : \mathrm{N}, x : X(n) \vdash h(n,x) : X(n+1)}{n : \mathrm{N} \vdash \mathrm{ind}_{g,h}(n) : X(n)}$$

  restricted to U_0-small type families $n : \mathrm{N} \vdash X(n) : \mathrm{U}_0$,
- larger universes U_α closed under all type constructors,
- and the rule

$$\Pi_{n:\mathrm{N}} X(n) : \mathrm{U}_{\max(1,\alpha)}$$

  for $X(n) : \mathrm{U}_\alpha$.

Then the definable terms

$$n : \mathrm{N} \vdash f(n) : \mathrm{N}$$

in T are exactly the primitive recursive functions in the standard model.

Motivation for Conservative Extension

A Priori Benefits

- Variants of primitive recursion are used as base theory for reverse mathematics in which theorems are encoded in a weak base system
- More expressive base system means less encoding
- Syntax is closer to proof assistants enabling formal verification

Potential Further Extensions
Univalent Type Theory

- Book HoTT hopeless – univalence does not compute
- Abstract syntax for Cubical TT well defined, but not modelled by Set
- Retracts of $\mathrm{y}_{\mathbb{N}}$ in cubical sets over p.r. base category remain of homotopy level zero

Potential Further Extensions

Reflection

- Can internally define syntax as inductive type and encoding $\lceil - \rceil : \mathrm{Term}_0 \to \mathrm{N}$
- Might be large, does not capture syntax up to judgemental equality
- Instead, could add type $\tilde{\mathrm{U}}_0 : \mathrm{tp}_0$ with $\mathrm{tm}_0 \tilde{\mathrm{U}}_0 = \mathrm{tm}_0 \mathrm{U}_0$
- Obtain code $u_0 : \mathrm{tm}_0 \mathrm{U}_0$ for all types tp_0
- No Π-types $\to$ no Girard's paradox

Potential Further Extensions
Inductive Constructions

- Large elimination principle
- Finitary inductive types and type families, finitary induction-recursion, lists, finitary function types
- U_0-fragment interpretable in arithmetic universes
- Modular semantic proof easy to adapt
- Important for convenience of programming

Synthetic Mathematics

State and Challenges

- Many settings for synthetic mathematics exist
  - Primitive recursion – restricted function types
  - Tait computability – Artin gluing, topological modalities
  - Algebraic geometry – $\mathrm{PSh}(k{-}\mathrm{Alg}^{\mathrm{op}})$
  - Topology – $\mathrm{Sh}(\mathbb{R}^n)$, $\mathrm{Sh}(2^{\mathbb{N}})$
  - Differential geometry, domain theory, probability theory, category theory, homotopy theory, condensed mathematics
- How to unify?
- How to mechanise externalisation?
- How to reflect?
- How to achieve base independence?
- How to deal with non-geometric sequents?

Synthetic Mathematics

Current Approaches

- MTT/MATT (Gratzer/Shulman) – no universal properties
- Fully geometric type theory (Vickers) – no function types
- Arithmetic type theory (Vickers) – base independece
- Synthetic topos theory (Uemura)
- Generalised (propositional) (∞)-geometric type theory (JSvB, Buchholtz, Williams)

Synthetic Mathematics

Principles Synthetic Mathemtics for Synthetic Mathematics

- Nullstellensatz (Blechschmidt) – Internally to $[\mathbb{T}]$, the universal model $U_{\mathbb{T}}$ satisfies a geometric* sequent σ iff the theory of $U_{\mathbb{T}}$-algebras proves σ
- Synthetic Quasicoherence / Blechschmidt duality:

$$A \simeq \mathbb{T}\text{-Alg}(A, U_{\mathbb{T}}) \to \mathbb{T}$$

- Quasicoherent induction (David Jaz Myers) – classifying toposes for homomorphisms satisfy directed path induction
- Directed univalence (Barton, Commelin)

$$\prod_{A,B:\mathrm{ODisc}} (A \to B) \simeq \sum_{\gamma:\mathbb{S}\to\mathrm{ODisc}} \gamma(\bot) = A \times \gamma(\top) = B$$

- Local Choice (Synthetic Stone Duality; Cherubini, Coquand, Geerlings, Moenclaey) – Surjections onto affine objects admit sections on a cover

Synthetic Mathematics

Generalised (propositional) (∞)-geometric type theory

- Consider sheaves on category of small-presented locales (resp. toposes) with open cover (resp. étalé) topology
- Have duality for internal small-presented frames A:

$$A \simeq \mathbb{S}\text{-Alg}(A, \mathbb{S}) \to \mathbb{S}$$

- Size issues